Name Solutions

February 9, 2012

ECE 311 Exam 1 Spring 2012

Closed Text and Notes

1) Be sure you have 10 pages and the additional pages of equations.

2) Write only on the question sheets. Show all your work. If you need more room for a particular problem, use the reverse side of the same page.

3) no calculators allowed

4) Write neatly, if your writing is illegible then print.

5) This exam is worth 100 points.

(9 pts) 1. Given the vectors $\mathbf{A} = 4\hat{\mathbf{a}}_x - 3\,\hat{\mathbf{a}}_z$ and $\mathbf{B} = 3\hat{\mathbf{a}}_x + 1\,\hat{\mathbf{a}}_y\; - 2\hat{\mathbf{a}}_z$

(3 pts) a) Find a unit vector in the direction of $\mathbf{A}$

$$\hat{a}_A = \frac{\vec{A}}{A} = \frac{\vec{A}}{\sqrt{\vec{A}\cdot\vec{A}}} = \frac{4\hat{a}_x - 3\hat{a}_z}{\sqrt{(4\hat{a}_x - 3\hat{a}_z)\cdot(4\hat{a}_x - 3\hat{a}_z)}} = \frac{4\hat{a}_x - 3\hat{a}_z}{\sqrt{16+9}}$$

$$\hat{a}_A = \frac{4}{5}\hat{a}_x - \frac{3}{5}\hat{a}_z$$

(3 pts) b) Find $\mathbf{A}\bullet\mathbf{B}$

$$\vec{A}\cdot\vec{B} = (4\hat{a}_x - 3\hat{a}_z)\cdot(3\hat{a}_x + 1\hat{a}_y - 2\hat{a}_z)$$

$$= 12 + 6$$

$$= 18$$

(3 pts) c) Find $\mathbf{A}\times\mathbf{B}$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ 4 & 0 & -3 \\ 3 & 1 & -2 \end{vmatrix}$$

$$= \hat{a}_x(0+3) - \hat{a}_y(-8+9) + \hat{a}_z(4-0)$$

$$= 3\hat{a}_x - 1\,\hat{a}_y + 4\hat{a}_z$$

(8 pts) 2. Identify the intersections of the following surfaces,

(a) $x = 1$ m, $y = -1$ m, and $z = -1$ m

point

(b) $r = 5$ m and $\theta = \frac{\pi}{2}$

circle

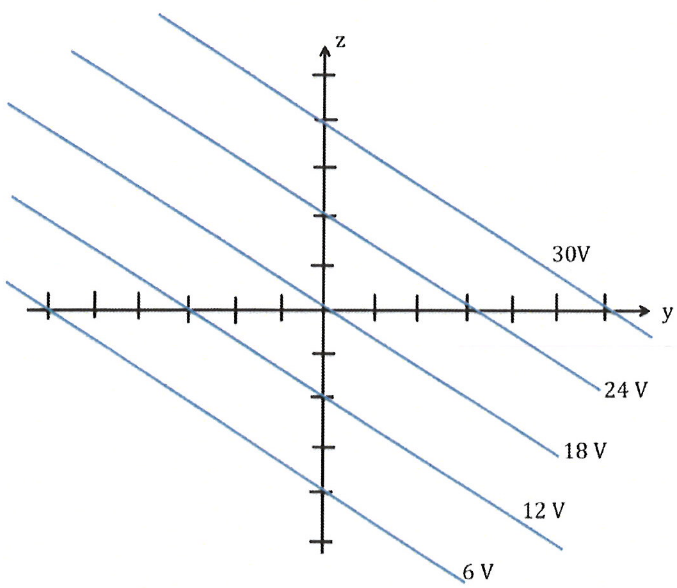
(c) $\rho = 1$m and $z = 3$ m.

circle

(d) $\theta = \frac{\pi}{2}$ and $\phi = \frac{\pi}{2}$

semi-infinite line with one end at the origin

(9 pts) 3. All of space is filled with infinite planes of equipotential that are parallel to the x-axis and spaced as shown. The ticks on the x- and y-axis represent 1 m steps. The potential step between the shown equipotential surfaces is 10 V. Determine the electric field intensity.



$$E_x = -\frac{dV}{dx} = 0$$

$$E_y = -\frac{dV}{dy} = -\frac{(24V - 18V)}{(3m - 0m)} = -\frac{6V}{3m} = -2\frac{V}{m}$$

$$E = -\frac{dV}{dz} = -\frac{(24V - 18V)}{(2m - 0m)} = -\frac{6V}{2m} = -3\frac{V}{m}$$

$$\vec{E} = (-2\hat{a}_y - 3\hat{a}_z)\frac{V}{m}$$

(6 pts) 4. A -5 μC charge is placed at $(\rho=1\,\text{m}, \phi=1.5\pi, z=3\text{m})$ and experiences a force of $\mathbf{F} = -15\hat{\mathbf{a}}_\rho + 10\hat{\mathbf{a}}_\phi + 25\hat{\mathbf{a}}_z\ N$. What is the electric field intensity at $(\rho=1\,\text{m}, \phi=1.5\pi, z=3\text{m})$?

$$\vec{E} = \frac{\vec{F}}{Q} = \frac{-15\hat{a}_\rho + 10\hat{a}_\phi + 25\hat{a}_z}{-5\times10^{-6}}\ \frac{N}{C}$$

$$\vec{E} = 3\times10^{6}\hat{a}_\rho - 2\times10^{6}\hat{a}_\phi - 5\times10^{6}\hat{a}_z\ \frac{N}{C}$$

$$\frac{N}{C} = \frac{Nm}{Cm} = \frac{J}{Cm} = \frac{V}{m}$$

$$\vec{E} = 3\times10^{6}\hat{a}_\rho - 2\times10^{6}\hat{a}_\phi - 5\times10^{6}\hat{a}_z\ \frac{V}{m}$$

(5 pts) 5. The z = 0 plane is an infinite plane of charge density 2 nC/m^2. Which one of the following statements best describes the equipotential surfaces caused by this infinite plane of charge?

A) The equipotential surfaces are planes extending radially outward from the z-axis.

B) The equipotential surfaces are infinite planes parallel to the z = 0 plane.

C) The equipotential surfaces are concentric cubes with the origin at the center.

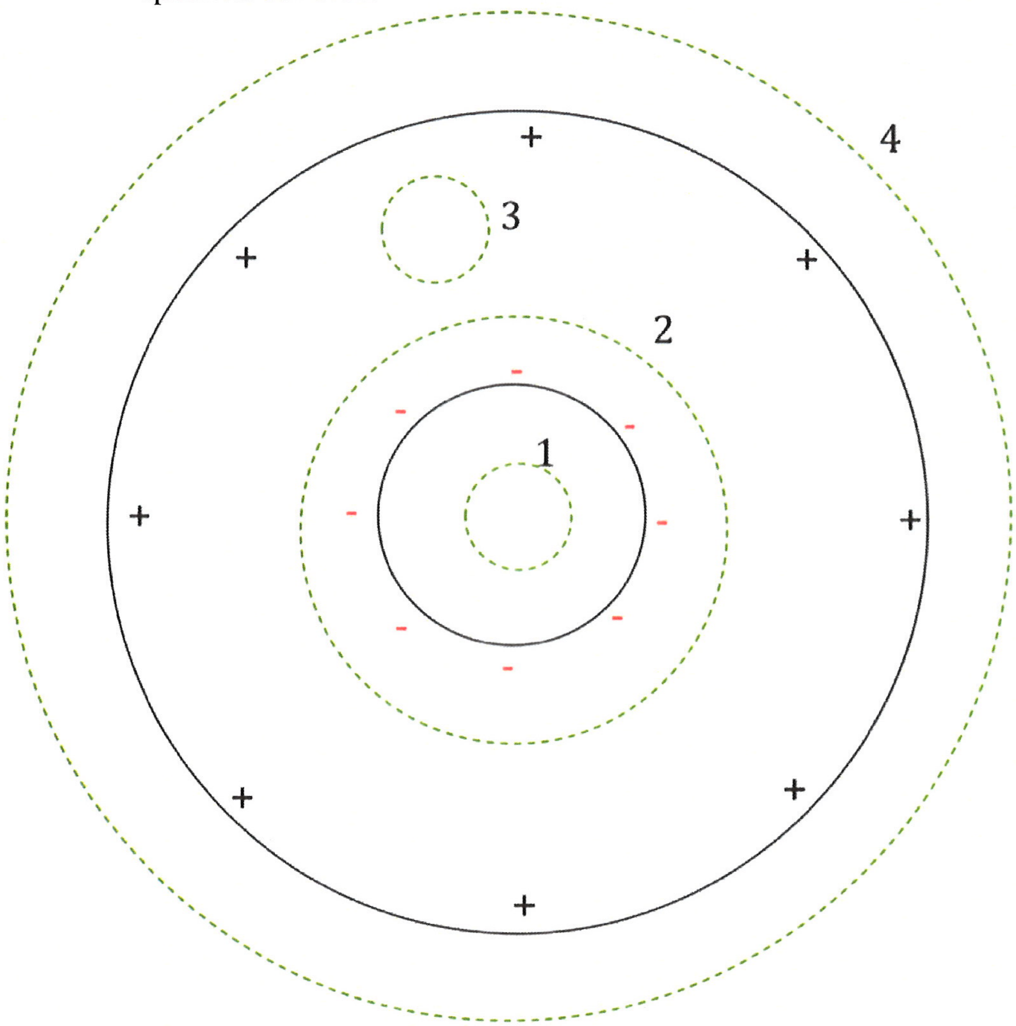
D) The equipotential surfaces are infinite planes perpendicular to the z = 0 plane.

E) The equipotential surfaces are concentric cylinders with the z-axis at the center.

(4 pts) 6. Suppose a uniform electric field exists in the room in which you are taking this exam such that the electric field lines are horizontal and at right angles to one wall. As you walk toward the wall from which the electric field lines emerge into the room, (so if you were a positive charge you would be pushing against the force due to the electric field) are you walking toward

a) points of the same potential, you are walking along an equipotential line.
b) points of lower potential.
c) points of higher potential.

(8 pts) 7. There are two nested co-centric spherical conducting shells as shown. The inner sphere his a total charge of -8 C and the outer sphere has a total charge of +8C. The dashed lines represents spherical surfaces of integration. Anything inside a dashed line is completely contained inside the spherical region represented by the dashed line. Determine the following integrals over the spherical surfaces.



$\oint \mathbf{D} \bullet \mathbf{dS}$ over surface 1 = 0

$\oint \mathbf{D} \bullet \mathbf{dS}$ over surface 3 = 0

$\oint \mathbf{D} \bullet \mathbf{dS}$ over surface 2 = -8C

$\oint \mathbf{D} \bullet \mathbf{dS}$ over surface 4 = 0

(20 pts) 8. Given that,

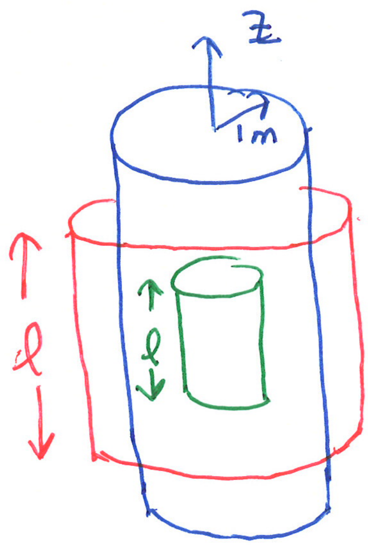
$$\rho_v = \frac{10^{-9}}{36\pi}\frac{C}{m^3} \quad \text{for} \quad \rho \le 1\text{m}$$

$$= 0 \quad \text{for} \quad \rho > 1\text{m}$$

Find the electric field intensity everywhere.

From the symmetry the electric flux density has the form

$$\vec{D} = D_\rho(\rho)\,\hat{a}_\rho$$



Gauss' law has to be applied in two regions, $\rho \le 1$m and $\rho > 1$m

$\rho \le 1$m

$$\oint \vec{D}\cdot\vec{ds} = Q_{encl.} = \rho_v(\pi\rho^2)\ell$$

$$D_\rho(\rho)(2\pi\rho)\ell = \rho_v(\pi\rho^2)\ell$$

$$D_\rho(\rho) = \frac{\rho_v\,\rho}{2}$$

$$E_\rho(\rho) = \frac{D_\rho(\rho)}{\epsilon_0} = \frac{\rho_v\,\rho}{2\epsilon_0} = \frac{\left(\frac{10^{-9}}{36\pi}\right)\rho}{2\left(\frac{10^{-9}}{36\pi}\right)} = \frac{\rho}{2}$$

$\rho > 1$m

$$\oint D\cdot ds = Q_{encl} = \rho_v(\pi(1)^2)\ell$$

$$D_\rho(\rho)(2\pi\rho)\ell = \rho_v(\pi)\ell$$

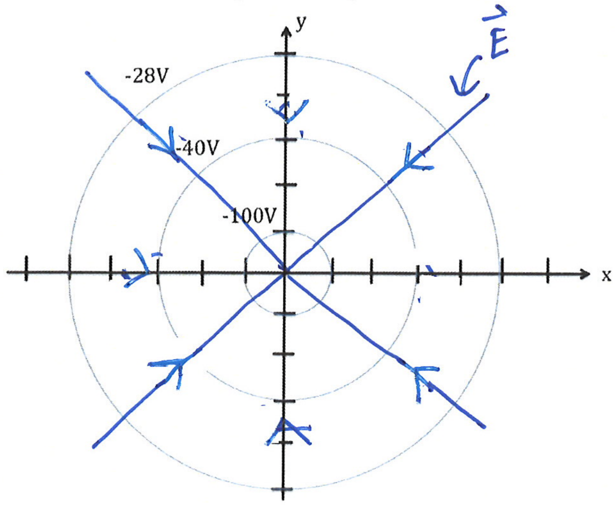
$$D_\rho(\rho) = \frac{\rho_v}{2\rho} \qquad E_\rho(\rho) = \frac{D_\rho(\rho)}{\epsilon_0} = \frac{\rho_v}{2\epsilon_0\rho} = \frac{1}{2\rho}$$

So,

$$\vec{E}(\rho) = \frac{\rho}{2}\hat{a}_\rho \quad \text{for} \quad \rho \le 1\text{m}$$

$$\frac{1}{2\rho}\hat{a}_\rho \quad \text{for} \quad \rho > 1\text{m}$$

(20 pts) 9. Shown are some equipotential surfaces caused by a point charge at the origin. Each tick on the x- and y-axis represent 1 m.



The electric field intensity caused by a point charge Q at the origin is

$$\vec{E} = \frac{Q}{4\pi\epsilon_o r^2}\hat{a}_r$$

(10 pts) a) Determine the value of the point charge at the origin.

$$V(1) - V(3) = -\int_{3m}^{1m} \vec{E}\cdot d\vec{l} = -\int_{3m}^{1m} \frac{Q}{4\pi\epsilon_o r^2}\hat{a}_r \cdot (dr\hat{a}_r + r\,d\theta\,\hat{a}_\theta + r\sin\theta\,d\phi\,\hat{a}_\phi)$$

$$(-100V) - (-40V) = -\int_{3m}^{1m} \frac{Q}{4\pi\epsilon_o r^2}dr = \frac{Q}{4\pi\epsilon_o r}\Big|_{3m}^{1m} = \frac{Q}{4\pi\epsilon_o}\left[\frac{1}{1m} - \frac{1}{3m}\right]$$

$$-60V = \frac{Q}{4\pi\epsilon_o}\frac{2}{3m} = \frac{Q}{2\pi\left(\frac{10^{-9}}{36\pi}\frac{F}{m}\right)}\frac{1}{3m} = \frac{6Q}{10^{-9}F}$$

$$Q = (-10V)(10^{-9}F) = -10\times10^{-9}C = -10\,nC$$

(5 pts) b) Determine $V(\infty)$

$$V(r) - V(\infty) = -\int_{\infty}^{r} \vec{E}\cdot d\vec{l} = \int_{\infty}^{r} \frac{10\times10^{-9}C}{4\pi\epsilon_o r^2}dr = -\frac{10\times10^{-9}C}{4\pi\epsilon_o r}\Big|_{\infty}^{r}$$

$$V(r) - V(\infty) = -\frac{10\times10^{-9}C}{4\pi\epsilon_o r}$$

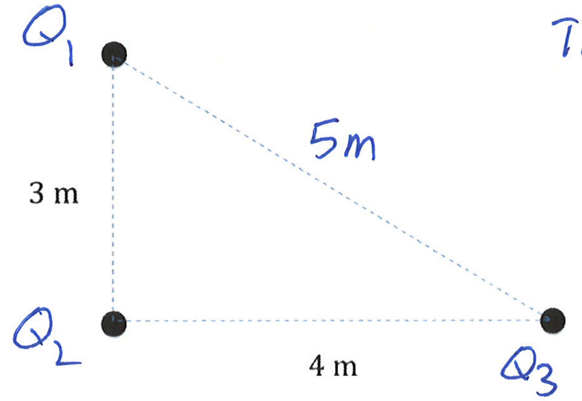
$$V(1m) - V(\infty) = -100V - V(\infty) = -\frac{10\times10^{-9}C}{4\pi\left(\frac{10^{-9}}{36\pi}\frac{F}{m}\right)(1m)} = -90V$$

$$V(\infty) = -10V$$

(5 pts) c) on the figure indicate the electric field intensity. (Just a qualitative sketch.)

(10 pts) 10. Three $\sqrt{\frac{10^{-9}}{9}}$ C point charges are located at the corners of a right triangle as shown.

Determine the energy stored in this charge arrangement.



To determine the energy stored bring in the three charges one at a time from infinity. Determine how much work this takes and this is the stored energy.

First bring in Q_1. This takes no work since there are no other charges present so $W_1 = 0$

Now bring in Q_2. This takes work because there is an electric field caused by Q_1

$$W_2 = Q_2 \frac{Q_1}{4\pi\epsilon_0 (3m)} = \frac{(10^{-9}/9)\,C^2}{4\pi\left(\frac{10^{-9}}{36\pi}\frac{F}{m}\right)(3m)} = \frac{1}{3} J$$

Now bring in Q_3. This takes work because there is an electric field caused by Q_1 and Q_2. We can use superposition and determine the work to bring in Q_3 with just Q_1 present and then with just Q_2 present and add.

$$W_3 = Q_3 \frac{Q_1}{4\pi\epsilon_0 (5m)} + Q_3 \frac{Q_2}{4\pi\epsilon_0 (4m)} = \frac{(10^{-9}/9)C^2}{4\pi\left(\frac{10^{-9}}{36\pi}\frac{F}{m}\right)(5m)} + \frac{(10^{-9}/9)C^2}{4\pi\left(\frac{10^{-9}}{36\pi}\frac{F}{m}\right)(4m)}$$

$$W_3 = \frac{1}{5} J + \frac{1}{4} J$$

$$\text{Energy Stores} = W_1 + W_2 + W_3 = 0 + \frac{1}{3} J + \frac{1}{5} J + \frac{1}{4} J = 0.783 \text{ J}$$